GR5065 Assignment 1

Due on Thursday, February 1, 2018 by 4PM

1 Expectations in Bowling

The tenth (final) frame of bowling works a bit differently than the previous nine frames.

1. If you get a strike (knock down all 10 pins) on your first roll, then you get two additional rolls. In that case, your second roll will have all 10 pins available again. If you get another strike, then on your third roll, there will be 10 pins available again. If you do not get a strike on your second roll, then the remaining pins are available to be knocked down on the third roll.
2. If you do not get a strike on your first roll of the tenth frame but do knock down all the remaining pins on your second roll (i.e. you get a spare), then you are given one additional roll with all 10 pins set back upright.
3. If you do not knock down all 10 pins by your second roll, then you do not get any additional rolls.

Use R to compute (without random number simulation) the expectation of the sum of the number of pins knocked down in the tenth frame of bowling if the Probability Mass Function (PMF) is the same as the one we used the first week:

```r
# computes the x-th Fibonacci number without recursion and with vectorization
F <- function(x) {
  stopifnot(is.numeric(x), all(x == as.integer(x)))
  sqrt_5 <- sqrt(5) # defined once, used twice
  golden_ratio <- (1 + sqrt_5) / 2
  return(round(golden_ratio ^ (x + 1) / sqrt_5))
}
# probability of knocking down x out of n pins
Pr <- function(x, n = 10) return(ifelse(x > n, 0, F(x) / (-1 + F(n + 2))))

Omega <- 0:10 # 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
```

You do not need to derive the answer algebraically.

Hint: Work out the *conditional* expectation of the number of pins knocked down in the tenth frame for each of the three cases mentioned above and weight these three cases by the probability of their occurrence to come up with the total expectation.

2 Probability in Poker

This question refers to the following hand of poker

https://youtu.be/PbmkJkBdxW0

If you do not know the rules of this game, you can read about them here

https://en.wikipedia.org/wiki/Texas_hold_%27em

Basically the two (face down) cards in your hand can be combined with any three (face up) cards that are in the middle to form a five card poker hand and there are several opportunities to fold or to bet at least as much as the other player(s).

Throughout you should assume that every card that a player cannot see has an equal chance to be any card that is remaining in the deck (after eliminating any cards that a player can see).

Ignore the commentators in the video, who are mostly making (bad) jokes and otherwise providing irrelevant observations to fill time in what they think will be a boring hand.

2.1 Pair Probability

Cary Katz (wearing the black baseball cap) is dealt the Ace of spades and the Ace of hearts.

```
* What is the probability of being dealt any two Aces?
* What is the probability of being dealt any pair of cards that have the same value?
```

Use algebra and show your work. Then, use simulations in R to check your work, which will somehow entail calling the `sample` function on `deck` where `deck` is

```r
deck <- rep(c("A", "K", "Q", "J", 10:2), each = 4)
```

2.2 Betting First

Before the hand is dealt, it is reasonable to assume each of the six players has a $\frac{1}{6}$ chance to win the hand. However, there is a substantial disadvantage to being the first person who has to decide whether to bet or fold. In terms of Bayesian probability theory (i.e. we are not looking for numbers here), why is there a disadvantage to being the first person who has to decide whether to bet or fold?

2.3 Decision Theory

As a result, when you have to decide to bet or fold first and there are five other players who are waiting for you to decide, many poker books say that you should only bet when your two (face down) cards are among the best 15% or so of possible hands (i.e. pairs, Ace-King, King-Queen, other adjacent cards that also have the same suit, etc.). How can we use the slide on Decision Theory from Tuesday to make sense of this advice?

2.4 Pair of Pairs

Since a pair of Aces is the very best hand you can be dealt, Cary Katz bets a standard amount for someone at that stage of a poker tournament. The next four players all have well below-average hands and quickly fold. Connor Drinan (wearing sunglasses) is dealt the Ace of diamonds and the Ace of clubs. What is the probability that two players at this table are both dealt a pair of Aces? Use algebra and show your work. Do not do random number simulations.

2.5 Bluffing

When the player that goes first bets and the next players all fold, the person that goes last will almost always either fold or "call" (i.e. bet the same amount as the first player and move on to see the first three cards in the middle). Since Connor Drinan knows that Cary Katz has among the top 15% of hands, it only makes sense to "raise" (i.e. bet more than the first player) when he has among the top, say, 5% of hands or when he is "bluffing" with a decent but considerably worse hand. In terms of Bayesian probability theory (i.e. we are not looking for numbers here), why is bluffing sometimes an advantageous strategy?

2.6 Chopping

Connor Drinan raises to a little more than a half million in chips, Cary Katz raises to two million, Drinan raises for all of his remaining chips and Katz calls, which was essentially an inevitable result when both players are dealt a pair of Aces. At about 2:10 in the video, Katz says (knowing that it is against the rules of poker) "Can we just chop?" meaning to give half the pot of $10,050,000 in chips to each player and move on to the next hand, which is what would happen eventually anyway unless one player makes a "flush" by having at least four of the five cards in the middle be from the same suit. As it says in the bottom left of the video, each player has a probability of 0.02 to make a flush and there is a 0.96 probability that neither player makes a flush, in which case their hands will result in a chopped pot.

At this point, what number is the expected gain or loss (in terms of chips) for both players? At this point, what number is the standard deviation (in terms of chips) for both players? In light of this, why does Katz wish they could chop the pot without even revealing the five cards in the middle?

2.7 Kings

At about 2:15 in the video, Drinan (quietly) says that he thought Katz "has to have (a pair of) Kings". Assume, quite plausibly, that in that situation, Katz would only raise to two million in chips if he had a pair of Aces, a pair of Kings, or an Ace and a King. From Drinan's perspective when he goes all in, what is the probability that Katz has a pair of Kings?

2.8 Free Roll

At about 2:32 in the video, Katz happily says he has a "four percent" chance to win because the second and third card in the middle are hearts and Katz has the Ace of hearts in his hand, leaving open the possibility that the last two cards in the middle will both be hearts and Katz will end up with a flush rather than merely a pair of Aces. However, the bottom left of the television screen says that Katz has a 5% chance of winning the pot. Which is closer to correct? Show your work, but you can use R to compute the probability.

2.9 Side Commentary

At about 2:53 in the video, Antonio Esfandiari says "I feel like it might (be another heart next)." Is this a Bayesian statement? Why or why not?

2.10 (In)dependence

Katz ultimately makes a flush and wins the entire pot. Is the event that a player is dealt two Aces independent of the event that a player makes a flush? Why?

2.11 History

At about 3:36 in the video, the (long-time) announcer speculates that the hand "might be the worst beat (luck) in the history of tournament poker". What is the pre-hand probability that two players are both dealt Aces and one player then makes a flush?

3 Bernoulli and Poisson

Suppose that X is a random variable that is the sum of n independent Bernoulli random variables with success probability $\pi = \frac{2}{3}$, but that n is a Poisson distributed random variable with expectation $\mu = \sqrt{7}$. What number is $\Pr\left(X = 4 | \, \mu = \sqrt{7}, \pi = \frac{2}{3}\right)$ and how did you arrive at it?

4 Frequentism

Describe one example in the social sciences (or whatever subfield you are most familiar with) where the frequentist view of statistics is plausibly appropriate, in the sense that someone might seek to ensure that an estimator has some useful and known property in the long run as the number of repeated samples of size N approaches infinity.

For example, each month the Bureau of Labor Statistics in the United States conducts the Current Population Survey, which asks people in $60,000$ households if they have a job or else are actively looking for a job. Then, the unemployment rate of the United States as a whole is estimated as

$$U = 100 \times \frac{\text{seeking job}}{\text{seeking job} + \text{have job}}.$$

Presumably, one could construct a 95% confidence interval estimator, (a, b), such that in 19 out of 20 months, we should expect that the true unemployment rate in the United States at that time (U_t) is contained in the estimated interval (a_t, b_t).